General Instructions

This examination is closed-book, and consists of three questions. Answer all three as clearly and concisely as you are able. Use of the internet and/or mobile devices is not permitted.
Question 1. (20 points total)

Consider the Prize Collecting TSP: we have $n$ cities, $m$ arcs in the underlying directed graph, the travel cost from city $i$ to city $j$ is $c_{ij}$ and there is a prize of value $p_i$ at city $i$. The salesman wants to visit some cities, and return to the starting city 1. (So he must visit city 1.) We have that $c_{ij} \geq 0$ and $p_i > 0$ for all $(i, j)$ and for all $i$.

If he visits a city $i$ then he gets to pick up the prize at that city. He wants to maximize his profit, which is

- The sum of the prizes he collects, minus the total travel cost.

1. (10 points) Formulate this problem with $O(n + m)$ variables, and constraints.

2. (5 points) In addition to the above, add an MIP constraint to force:

- If city 2 and city 5 are both visited, then city 2 must be visited before city 5.

The total number of variables and constraints still should be $O(n + m)$.

3. (5 points) For this part, ignore part 2. Formulate the following using an MIP constraint. The total number of variables and constraints still should be $O(n + m)$.

- If both city 2 and city 5 are visited, and city 5 is visited before city 2, then we must pay a penalty of $C$ dollars, where $C$ is a positive constant.
Question 2. (21 points total)
This question consists of three parts.

1. (7 points) Let \( A \in \mathbb{R}^{m \times n} \) be given. Prove statements (a) and (b) below are equivalent.
   (a) There exists \( x \in \mathbb{R}^n \) such that \( Ax \geq 0 \) and \( Ax \neq 0 \).
   (b) There does not exist \( y \in \mathbb{R}^m \) such that \( y^T A = 0 \) and \( y > 0 \).

2. (7 points) Now consider statement (b’) below:
   (b’) There does not exist \( y \in \mathbb{R}^m \) such that \( y^T A = 0 \), \( y \geq 0 \) and \( y \neq 0 \).
   Use an example to show that statements (a) and (b’) are not equivalent.

3. (7 points) Fill in the blank below:
   (a’) There exists \( x \in \mathbb{R}^n \) such that \( \underline{\text{------------------------}} \)
   such that statement (a’) is equivalent to statement (b’). Justify your answer.
Question 3. (10 points total)

Consider a polyhedron $P$ defined as follows.

$$P = \left\{ x \in \mathbb{R}^3 \left| \begin{bmatrix} -1 & -6 & 1 \\ -1 & -2 & 7 \\ 0 & 3 & -10 \\ -6 & -11 & -2 \\ 1 & 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 4 \\ -7 \\ 0 \\ 6 \end{bmatrix} \right. \right\}$$

For each of the following $x^*$, decide if $x^*$ is an extreme point of the polyhedron $P$. If it is, find a vector $c \in \mathbb{R}^3$ such that $x^*$ is the unique maximizer of $c^T x$ among all points $x \in P$. Otherwise, find two disjoint points $x_1$ and $x_2$ in $P$, so that $x^*$ is the midpoint of $x_1$ and $x_2$.

1. $x^* = (1, 1, 1)$.
2. $x^* = (7, 0, 1)$. 