Stor 635 questions (2015/16 CWE)

Problem 1. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent classes of events, each closed under intersections. Let $\mathcal{B}_1 = \sigma(\mathcal{A}_1)$ and $\mathcal{B}_2 = \sigma(\mathcal{A}_2)$. Show that $\mathcal{B}_1$ and $\mathcal{B}_2$ are also independent classes of events.

Problem 2. (i) Show that convergence in probability implies convergence in distribution, that is, if $\xi_n \xrightarrow{P} \xi$, then $\xi_n \xrightarrow{d} \xi$. (ii) Show the converse if the limit is a constant random variable, that is, if $\xi_n \xrightarrow{d} \xi$ and $\xi = c$ a.s. (for a constant c), then $\xi_n \xrightarrow{P} \xi$.

Problem 3. Let

$$\xi_n = \begin{cases} \frac{1}{n}, & \text{with prob. } p_n, \\ -\frac{1}{n}, & \text{with prob. } 1-p_n, \end{cases} \qquad n \geq 1,$$

be independent random variables. Find necessary and sufficient conditions for the convergence of the series $\sum_{n=1}^{\infty} \xi_n$ in terms of the sequence $\{p_n\}_{n\geq 1}$.

Problem 4. (i) Let ξ be a random variable with mean 0, finite variance σ^2, distribution function F and characteristic function ϕ. Show that ϕ can be written as

$$\phi(t) = 1 - \frac{1}{2}\sigma^2 t^2 \psi(t),$$

where ψ is the characteristic function of the probability density function

$$g(x) = \begin{cases} \frac{\sigma^2}{2}\int_x^{\infty}(1-F(u))du, & \text{if } x \geq 0, \\ \frac{\sigma^2}{2}\int_{-\infty}^{x} F(u)du, & \text{if } x < 0. \end{cases}$$

(ii) Deduce from (i) the elementary form of the central limit theorem, that is, if ξ_n, $n \geq 1$, are i.i.d. random variables with mean 0 and variance σ^2, then $\frac{1}{\sqrt{n}\sigma}\sum_{k=1}^{n}\xi_k$ converges in distribution to the standard normal random variable.

Problem 5. Let $\{\xi_i^n, i, n \geq 1\}$ be i.i.d. nonnegative, integer-valued r.v.'s. Consider a branching process $\{Z_n\}_{n\geq 1}$ defined by $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} 0 & \text{if } Z_n = 0, \\ \xi_1^{n+1} + \ldots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0. \end{cases}$$

Let $\mathcal{F}_n = \sigma\{\xi_i^m, i \geq 1, m \leq n\}$ and suppose $\mu = \mathbb{E}\xi_i^m \in (0, \infty)$. (i) If $\mu > 1$ and $s_0 \in [0, 1)$ is the unique point satisfying $\phi(s_0) = s_0$ for a probability generating function $\phi(s) = \sum_{k=0}^{\infty}\mathbb{P}(\xi_i^n = k)s^k$, show that $\{s_0^{Z_n}, \mathcal{F}_n\}$ is a martingale. (ii) Under the assumptions and notation of (i), use the martingale convergence results to show that the extinction probability $\mathbb{P}(Z_n = 0 \text{ for some } n)$ equals s_0.