STOR 635, CWE 2016-17

All four questions carry equal weight.

Problem 1. (12.5) Show that the following two are equivalent for a class $\mathscr{H}$ of integrable real random variables on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$.

(i) $\mathscr{H}$ is uniformly integrable.
(ii) The class $\mathscr{H}$ is L^1 bounded and for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup_{f \in \mathscr{H}} \int_A |f| d\mathbf{P} \leq \varepsilon \text{ whenever } A \in \mathscr{F} \text{ satisfies } \mathbf{P}(A) < \delta.$$

Problem 2. (12.5) Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space and let $\mathscr{G}$ be a sub σ-field of $\mathscr{F}$. Let X, X_n be integrable random variables on $(\Omega, \mathscr{F}, \mathbf{P})$. Prove the following Monotone Convergence theorem for conditional expectations: If $X_n \geq 0$ and $X_n \uparrow X$ a.s., then $\mathbf{E}(X_n \mid \mathscr{G}) \uparrow \mathbf{E}(X \mid \mathscr{G})$ a.s.

Problem 3. (12.5) Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space on which we are given a filtration $\{\mathscr{F}_n\}_{n \geq 0}$. Let $\{X_n\}$ be a square integrable $\{\mathscr{F}_n\}$-martingale. Show that the following are equivalent:

(i) $\sup_{n \in \mathbb{N}_0} \mathbf{E}|X_n|^2 < \infty$.
(ii) $\lim_{n \to \infty} \mathbf{E}[\langle X_n \rangle] < \infty$.
(iii) X_n converges in L^2.
(iv) X_n converges a.s and in L^2.

[Hint: You may use Doob's L^p inequality without proof.]

Problem 4. (12.5) Let E be a Polish space and let $\{X_n\}_{n \in \mathbb{N}_0}$ be a E valued Markov chain with distributions $\{\mathbf{P}_x\}_{x \in E}$ on $(\Omega, \mathscr{F})$. Let τ be a finite stopping time with respect to $\{\mathscr{F}_n\}$ where $\mathscr{F}_n = \sigma\{X_0, \ldots, X_n\}$. Prove the following strong Markov property: For any bounded measurable map $F : \mathbb{N}_0 \times E^\infty \to \mathbb{R}$

$$\mathbf{E}_x \left(F(\tau, X_{\tau+\cdot}) \mid \mathscr{F}_\tau \right) = G(\tau, X_\tau)$$

where $G(m,x) \doteq \mathbf{E}_x(F(m,X))$ for $(m,x) \in \mathbb{N}_0 \times E$.