STOR641 - Comprehensive Written Exam - August 2017

This test consists of three questions. The relative weights are specified following each part. You are NOT allowed to refer to any notes, books etc.

Good luck!

1− Consider a service system where customers arrive according to a Poisson process with rate λ per minute.

(a) Suppose that there are currently no customers in the system and the server will start working as soon as there are four customers in the system. What is the expected time until the server starts working?**(10 points)**

(b) Suppose now that instead of observing the queue continuously the server checks the queue at one minute intervals and as soon as she sees four customers in the system, she starts serving them. What is probability distribution for the time until the server starts working given that there are currently no customers in the system?**(10 points)**

(c) Suppose that the server works as described in part (b) and there are no customers initially in the system. Suppose also that the server starts working at time t. If there were three customers in the system at time $t-1$ and there were two customers at time $t-2$, what is the probability distribution for the interarrival time between the third and fourth customers?**(15 points)**

2− Consider two frogs moving on an infinite size one-dimensional lattice with lattice points numbered as $\{1, 2, 3, \dots\}$ going from left to right. At each time period, Frog Albert either moves 1 point to the right with an independent probability of p or stays in the same location with probability $1-p$. In a similar fashion, Frog Barbara either moves 1 point to the left with an independent probability p or stays in the same location with probability $1-p$. Suppose that Albert is currently on point 3 and Barbara is currently on point 7.

(a) What is the probability that Albert and Barbara will be in the same position at some point?**(10 points)**

(b) What is the expected time until Albert and Barbara are either in the same location or Barbara is to the left of Albert? **(10 points)**

3– Consider a service system operating as follows. Customers are either of type-1 or type-2. Type-1 customers first receive service from server-1 and type-2 customers first receive from server-2. However, a type-1 customer, after receiving service from server-1, would also need service from server-2 with probability α independently of everything else. Similarly, a type-2 customer, after receiving service from server-2, would also need service from server-1 with probability β independently of everything else. There is no waiting space in this system and therefore, an arriving type-i customer who finds server-i either busy serving another customer or blocked is lost. (The following will explain how a server can be blocked.)

When server-1 finishes serving a customer who also needs service from server-2, if server-2 is free, the customer immediately starts receiving service from server-2 and server-1 becomes available for new customers; however, if server-2 is busy serving another customer, the customer who has just finished service with server-1 waits until server-2 becomes available and server-1 remains blocked during that time and cannot serve any other customer. Similarly, when server-2 finishes serving a customer who also needs service from server-1, if server-1 is free, the customer immediately starts receiving service from server-1 and server-2 becomes available for new customers; however, if server-1 is busy serving another customer, the customer who has just finished service with server-2 waits until server-1 becomes available and server-2 remains blocked during that time and cannot serve any other customer. (Note that the two servers cannot both be blocked at the same time as they would immediately switch jobs once they are done with their respective services.)

Service times of server-i, for $i = 1, 2$, are iid exponentially distributed with rate μ_i and arrival times of type-i customers follow an independent Poisson process with rate λ_i.

(a) Model this system as a continuous-time Markov chain (CTMC). Clearly describe the state space and give the transition probabilities.**(20 points)**

(b) Let p_j denote the steady-state probability that the system is in state j. Write down the balance equation only for the state that represents the empty system. **(5 points)**

(c) Give an expression for the long-run fraction of time one of the servers is blocked. Do NOT solve the balance equations. Simply give your answer using p_js. **(10 points)**

(d) Give an expression for the long-run fraction of customers who are lost because of a blocked server. Do NOT solve the balance equations. Simply give your answer using p_js. **(10 points)**