STOR641 - Comprehensive Written Exam - August 2019

This test consists of three questions. The relative weights are specified following each part. You are NOT allowed to refer to any notes, books etc.

Good luck!

1− Consider a service system with two servers serving three types of customers. Each server has a space to hold one customer and if this space is occupied arriving customers will go away. Arrivals to Server 1 can be either of type 1 or type 3 while arrivals to Server 2 can be either of type 2 or type 3. However, while type 1 customers are served by Server 1 alone and type 2 customers are served by Server 2 alone, service of type 3 customers requires active involvement of both servers. Therefore, if a type 3 customer arrives to receive service from Server i ($i = 1, 2$), when serving the customer, Server i and Server $3 - i$ will both be busy. The service is non-preemptive however meaning that if Server $3 - i$ is busy with serving a customer at the time type 3 customer arrives to Server i, service will only start when Server $3 - i$ completes the service in progress. Note also that if a type 3 customer arrives at Server i, when in service that customer will only occupy the space for Server i. The space for Server $3 - i$ will be available for future customers. However, the service of customers who arrive while Server $3 - i$ is occupied with the service of a type 3 customer who is located at the space for Server i starts only after the service of type 3 customer is over. Type 1 and type 2 customers arrive according to independent Poisson processes with respective rates, λ_1 and λ_2. Type 3 customers arrive at Server 1 according to an independent Poisson process with rate λ_{31} and they arrive at Server 2 according to another independent Poisson process with rate λ_{32}. All service times are independent and exponentially distributed. The mean service time for type i ($i = 1, 2, 3$) customers is $1/\mu_i$.

(a) Model this system as a continuous-time Markov chain (CTMC). Clearly describe the state space $\mathcal{S}$ and give the transition rates.**(20 points)**

(b) For any $j \in \mathcal{S}$, let p_j denote the steady-state probability that the system is in state j. Write down the balance equation only for the state that represents the empty system. **(5 points)**

(c) What is the long-run rate with which type 3 jobs leave the system after receiving service? Give your answer using p_js. DO NOT SOLVE THE BALANCE EQUATIONS. **(10 points)**

(d) What fraction of arriving customers are turned away? Give your answer using p_js. DO NOT SOLVE THE BALANCE EQUATIONS. **(8 points)**

(e) Among the type-3 customers who receive service what fraction are served right away without waiting for any one of the servers to become available. Give your answer using p_js. DO NOT SOLVE THE BALANCE EQUATIONS. **(7 points)**

2− Suppose that female and male customers arrive at a store that is only open for three hours everyday according to independent Poisson processes with independent rates λ_f and λ_m per hour.

(a) Give an expression for the probability that the number of arrivals during the first two hours and the last two hours will both be two.**(9 points)**

(b) If three customers arrive during the whole three-hour period, what is the probability that none arrives during the first hour?**(8 points)**

(c) If five customers arrive during the whole three-hour period, what is the probability that three of them are female?**(8 points)**

3– Let $\{X_n, n \geq 0\}$ be a DTMC with state space $\mathcal{S} = \{0, 1, 2, \dots\}$ and transition probabilities given by $P_{0i} = (0.5)^i$ for $i \geq 1$ and $P_{i,i-1} = 1$ for $i \geq 1$. All other transitions have a probability of zero.

(a) Is $\{X_n, n \geq 0\}$ irreducible? Why or why not?**(5 points)**

(b) Is $\{X_n, n \geq 0\}$ periodic or aperiodic?**(5 points)**

(c) Is $\{X_n, n \geq 0\}$ transient, positive recurrent, or null-recurrent? Show why.**(10 points)**

(d) Let T denote the first time the DTMC visits state 2 and define $m_0 = E[T|X_0 = 0]$ What is m_0?**(5 points)**