STOR 642
Comprehensive Written Examination
9:00am-1:00pm, August 16, 2018

This test consists of two questions.
This is a closed book exam. No communication devices of any kind are allowed.
Explain your answers in detail.
The problem and subproblem weights are given in parentheses.

Problem 1. (50 points) Consider an inventory system that serves customers of two types, each demanding one item upon arrival. It uses a policy with three parameters (A, B, C), where A, B, C are integers such that $A > B > C > 0$. It operates as follows. Suppose $I(t)$ is the inventory on hand at time t. When a customer of type 1 arrives at time t and $I(t) > 0$, the customer demand is satisfied, otherwise the customer is turned away. If a customer of type 2 arrives at time t and $I(t) \geq C$, the customer demand is satisfied, otherwise the customer is turned away. There is no backlogging. When the inventory on hand drops below B, an order of size A is placed, and it arrives at time $t + L$, where L is the lead time. The policy allows only one outstanding order at a time.

Suppose the combined arrivals of the two types occur according to a renewal process with common inter-arrival time distribution $F(\cdot)$ and mean τ. Each arrival is of type 1 with probability α and type 2 with probability $\beta = 1 - \alpha$. The successive lead times are iid exponential random variables with mean $1/\mu$. Suppose the inventory has just reduced to $B - 1$ at time 0.

1. (10) Argue that $I(t)$ uniquely determines the number of outstanding orders (0 or 1) at time t. What is the state space of $\{I(t), t \geq 0\}$?
2. (20) Show that $\{I(t), t \geq 0\}$ is an SMP. Compute its kernel.
3. (10) Under what condition does the limiting distribution of $I(t)$ exist? Show how to compute it assuming it exists.
4. (10) Using this distribution compute the long run cost, assuming the following cost model: the holding cost is h dollars per item per unit time, the cost of losing a demand of type k is r_k, $k = 1, 2$.

Problem 2. (50 points) Suppose a seller has one item to sell to a single customer. The customer valuation of the item is a random variable V, whose actual value is known to the customer, but unknown to the seller. If the the seller puts the item on sale at price p, the customer will buy the item if $V \geq p$, and will not buy it otherwise. Suppose initially the seller believes that $V \sim U(0,1)$. If the seller offers the item at price p, and the customer buys it, the seller gets a revenue of p, and problem terminates. If the item does not sell at the set price, the seller knows that $V < p$, and (updating his belief about the distribution of V) can offer it again at a reduced price. The seller may do so until the item sells, or a total of N attempts are made, where N is a fixed positive integer. If the item does not sell after N attempts, the process ends, and the seller receives nothing. The seller's aim is to set the prices so as to maximize the expected revenue from the sale.

1. (10) Suppose there are k more attempts left, and the item has not sold yet, and the seller currently believes that $V \sim U(0,x)$ for some $x \in [0,1]$. (Thus, at the beginning, $k = N$ and $x = 1$.) Suppose the seller chooses to sell the item at price $p \in [0,x]$. What is the probability that the sale will go through? What is the probability that the sale will fail? If the sale fails, what is the updated belief of the seller about the distribution of V?

2. (10) Let $v_k(x)$ $(1 \leq k \leq N,\ 0 \leq x \leq 1)$ be the optimal expected revenue if there are k more selling attempts left, the item is not sold yet, and the current prior distribution of V is $U(0,x)$. The decision is to pick the price $p \in [0,x]$ at this stage. Derive a Bellman equation for $v_k(x)$. What is $v_0(x)$?

3. (10) Let $p_k(x)$ be the optimal price to use when there are k attempts left to go, and the current prior is $U(0,x)$. Compute $v_1(x),\ p_1(x)$ and $v_2(x),\ p_2(x)$.

4. (10) Show that $p_k(x) = \frac{kx}{k+1}$, and $v_k(x) = \frac{x}{2}\frac{k}{k+1},\ k \geq 1$.

5. (10) Suppose the initial valuation of the item is $U(0,1)$. Compute the optimal price r_n to use in period n if a maximum of N periods are available, and the first $n-1$ offers have been uusuccessful. Show that the maximum expected revenue is $\frac{1}{2}\frac{N}{N+1}$.