August 2019 Name:______________________

Comprehensive written exam – STOR654 Mathematical Statistics

All problem parts have equal weight. In budgeting your time expect that some part will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts. You do not have to prove the hint. Do not forget to split the time between both papers!

1. Let $X \sim \text{Binomial}(n, p)$. Evaluate the p-values and Bayes factors in parts (a) – (d) using $n = 10$ and $x = 1$.

   (a) Propose a p-value for testing $\mathcal{H}_0 : p \in [0, 1/3]$ vs. $\mathcal{H}_1 : p \in (1/3, 1]$.

   (b) Assuming that the prior is $p \sim U(0, 1)$, find the Bayes factor for testing $\mathcal{H}_0 : p \in [0, 1/3]$ vs. $\mathcal{H}_1 : p \in (1/3, 1]$.

   (c) Modify the prior to test $\mathcal{H}_0 : p = 1/3$ vs. $\mathcal{H}_1 : p \neq 1/3$ and compute the Bayes factor.

   (d) Propose a p-value for testing $\mathcal{H}_0 : p = 1/3$ vs. $\mathcal{H}_1 : p \neq 1/3$.

   (e) Show that for $0 < \epsilon < p$,

   $$P(X/n \leq \epsilon) \leq \exp\left\{ -n(1-\epsilon)\log\left(\frac{1-\epsilon}{1-p}\right) - n\epsilon\log\left(\frac{\epsilon}{p}\right)\right\}.$$

   Compare this upper bound and the actual probability in the case $n = 10$, $p = 1/3$, and $\epsilon = 0.1$.

2. Let X_i, $i = 1, \dots, n$ be i.i.d. $\Gamma(a, b)$, the gamma distribution with shape parameter a and scale parameter b.

   (a) **Prove** that the moment generating function is $M_{X_i}(t) = (1-bt)^{-a}$ for $t < b^{-1}$.

   (b) Find $\theta = EX_i^k$, $k \in \mathbb{N}$. Calculate the coefficient of variation $\frac{\text{sd}\, X_i}{EX_i}$.

   (c) What is the distribution of $Z = \frac{X_1}{\sum_{i=1}^n X_i}$? Carefully **justify** your answer!

   (d) Find $\hat{\theta}_{MM}$ the MM estimator of $\theta = ab$. Is it UMVUE? What is its MSE?

(e) Derive a 95% confidence interval for a.

3. Let X_i be independent, $X_i \sim \Gamma(i, b)$, $i = 1, \ldots, n$. (Use the same parametrization as above.)

   (a) Does the distribution of the random vector $\mathbf{X} = (X_1, \ldots, X_n)'$ belong to an exponential family?

   (b) Is the MLE of $1/b$ biased? If so, find its bias.