STOR 672
Comprehensive Written Exam
August 15, 2018

- This exam consists of 4 questions on 2 pages.
- The exam is closed book and closed notes.
- You are NOT allowed to use a calculator or a cell phone during the exam.
- Explain your answers in detail.

Problem 1. (25 points) Draw a flow chart of a discrete-event simulation code that is written in a general programming language and uses the next-event-time-advance approach. Be specific about the routines used and what is done in each routine.

Problem 2. (25 points) A team of analysts uses simulation to estimate W, which is the long-run average waiting time of patients at an emergency department of a hospital. Starting the simulation from an empty and idle state, they conduct a single simulation run to collect observations X_i, for $i = 1, \ldots, n$, where X_i represents the waiting time of the ith patient to arrive at the emergency department.

(a) The team first obtains a point estimator for W. For this purpose, they use the sample mean of all observations collected, i.e., $\bar{X}_n = \sum_{i=1}^{n} X_i/n$. Is this an unbiased estimator? Why or why not?

(b) The analysts recall from their simulation classes that they took at college that it is not sufficient to provide a point estimator but they also need to report some sort of a measure of sampling error. For that purpose, they decide to construct a 95% confidence interval on W using the data collected. For this purpose, they compute the sample variance, i.e., $S_n^2 = \sum_{i=1}^{n}(X_i - \bar{X}_n)^2/(n-1)$, and estimate the half-width of the confidence interval by $H = t_{n-1,0.975}\sqrt{S_n^2/n}$, where $t_{d,\alpha}$ is the α quantile of a t distribution with d degrees of freedom. Is this an unbiased estimator? Why or why not?

(c) The resulting 95% confidence interval reported to the hospital management was $\bar{X}_n \pm H$. What can you say about the coverage of this interval? Explain your answer.

Problem 3. (25 points) Consider $\{N(t), t \geq 0\}$, which is a non-stationary Poisson process with a non-negative and bounded rate function $\lambda(t)$. Let t_i denote the ith event time corresponding to this Poisson process.

(a) Provide an algorithm to generate a sample path of t_i's for $i = 1, \ldots, n$.

(b) Prove that the algorithm provided in part **(a)** works. Hint: You may want to use the following definition of a Poisson process. A counting process $\{N(t), t \geq 0\}$ is a Poisson process with rate function $\lambda(t)$ if and only if (i) $\{N(t), t \geq 0\}$ has independent increments, (ii) $N(0) = 0$ and

$$\begin{aligned}
\Pr\{N(t+h) - N(t) = 0\} &= 1 - \lambda(t)h + o(h), \\
\Pr\{N(t+h) - N(t) = 1\} &= \lambda(t)h, \\
\Pr\{N(t+h) - N(t) = j\} &= o(h), \text{ for } j \geq 2.
\end{aligned}$$

Problem 4. (25 points) Two friends Ann and Sue are trying to obtain the value of sum of two integrals by means of Monte Carlo simulation. More specifically, they would like to estimate $I_1 + I_2$, where $I_i = \int_0^1 g_i(x)dx$, by using a sequence of independent random numbers $\{U_j, j = 1, 2, \ldots, n\}$. Ann proposes that they let $h(x) = g_1(x) + g_2(x)$ and apply the Monte Carlo method to estimate $I = \int_0^1 h(x)dx$ using all n random numbers available. Sue thinks they should estimate I_1 and I_2 separately using half of the random numbers for one estimation and the remaining half for the other, and then add up the two estimates. (Assume that n is an even number.)

(a) Show that both Ann's and Sue's approaches will yield an unbiased estimator for $I_1 + I_2$.

(b) Which approach will yield an estimator with a smaller variance, Ann's or Sue's? Support your answer by comparing the variances of both estimators.