STOR 672
Comprehensive Written Exam
August 14, 2019

- This exam consists of 3 questions on 2 pages.
- This is a closed book exam.
- You are not allowed to use a calculator or a cell phone.
- Explain your answers in detail.

Problem 1. (35 points) Consider a queueing system with six parallel identical servers. Customers arrive to this queueing system according to a Poisson process and join a single queue. The service time for each customer is independent and identically distributed. Suppose that a single simulation run of this queueing system has been conducted to collect the following data: $\{N(t), 0 \leq t \leq T\}$, where $N(t)$ is the number of jobs in the system at time t and T is the total simulated time. This simulation was initialized at a state where all servers are busy but there are no other jobs waiting in the queue. Based on a prior warm-up period analysis, it is estimated that the stochastic process $\{N(t), t \geq 0\}$, which is initialized as described, reaches steady state at around $t = \tau$, where τ is much smaller than T.

(a) (12 points) Explain how you would apply the batch means method with b batches to construct a 95% confidence interval on the steady-state mean number of busy servers. Be specific about the upper and lower bounds of the confidence interval, i.e., express them as functions of $\{N(t), 0 \leq t \leq T\}$, T, τ, etc.

(b) (6 points) Name one method to check the validity of the confidence interval obtained in part **(a)** using batch means.

(c) (7 points) If the confidence interval obtained in part **(a)** passes the test for validity in part **(b)** but is too large for the desired precision, what should be done to reduce the confidence interval length (to about one-third of its size) without compromising its validity?

(d) (10 points) The simulated process $\{N(t), 0 \leq t \leq T\}$ enters state zero twenty times and $N(T) = 7$. Explain how the regenerative method can be used to provide a point estimator for the long-run average fraction of time the queue is empty. Be specific about the point estimator, i.e., express it in terms of $\{N(t), 0 \leq t \leq T\}$, T, etc.

Problem 2. (35 points) $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are two time-homogenous continuous-time stochastic processes defined on the same finite state space $\mathcal{S}$. For both processes, when the process leaves state $i \in \mathcal{S}$, it next enters state $j \in \mathcal{S}$ with some probability P_{ij}, where $P_{ii} = 0$ and $\sum_{j\in\mathcal{S}} P_{ij} = 1$. Each time $\{X(t)\}$ enters state $i \in \mathcal{S}$, the amount of time it spends in that state before making a transition into a different state is exponentially distributed with mean $\tau_i > 0$. On the other hand, the sojourn time of $\{Y(t)\}$ in state $i \in \mathcal{S}$ is uniformly distributed over $[\tau_i - \epsilon_i, \tau_i + \epsilon_i]$, where $0 < \epsilon_i < \tau_i$ for all $i \in \mathcal{S}$. For both processes, all sojourn times and transitions are independent. (Note that $\{X(t), t \geq 0\}$ is a continuous-time Markov chain and $\{Y(t), t \geq 0\}$ is a semi-Markov process.) In this question, we will design a simulation experiment to compare the fraction of time each process spends in state $k \in \mathcal{S}$ during $[0, T]$ if both processes just entered state k at time zero.

(a) (8 points) Provide an algorithm to simulate $\{X(t), 0 \leq t \leq T\}$.

(b) (8 points) Provide an algorithm to simulate $\{Y(t), 0 \leq t \leq T\}$.

(c) (7 points) Explain how you would synchronize random numbers used in the algorithms provided in parts **(a)** and **(b)** if we would like to use common random numbers.

(d) (12 points) Explain how you would construct a 95% confidence interval on the difference between the fraction of time these two processes spend in state k during $[0, T]$ using common random numbers. Define necessary notation to be specific about the upper and lower bounds of the confidence interval.

Problem 3. (30 points) Consider a random variable X that has a geometric distribution with parameter $p \in (0, 1)$, i.e., $\Pr\{X = i\} = p(1 - p)^i$ for $i = 0, 1, \ldots$.

(a) (11 points) Prove that the following is an exact algorithm that generates geometric random variates:

Step 1. Generate a random number U.

Step 2. Return $X = \lfloor \ln(U)/\ln(1 - p) \rfloor$, where $\lfloor x \rfloor$ is the floor function that returns the largest integer smaller than or equal to x.

(b) (11 points) Provide another exact algorithm for generating a geometric random variate.

(c) (8 points) Compare the algorithms provided in parts **(a)** and **(b)** in terms of their computational efficiency.