STOR 762
Comprehensive Written Exam
August 17, 2016

- This exam consists of 4 questions on 2 pages.
- The exam is closed book and closed notes.
- You are NOT allowed to use a calculator or a cell phone throughout the exam.
- Explain your answers in detail.

Problem 1. (25 points) Consider a queueing network with two stations in series, where jobs arrive and receive service at the first station, enter the second station to receive additional service, and then depart from the system. Each station has a single server and an infinite-capacity queue. The system is empty at time zero. The performance measures of interest are the average total number of jobs in the system and the average fraction of time both servers are busy during the first τ minutes. Suppose that you are asked to write a simulation code using the next-event time-advance approach in a general purpose programming language to estimate the performance measures of interest for this queueing network. Answer the following questions.

(a) (5 pts.) Define the state of the system to be used in your simulation code.

(b) (5 pts.) What are the events associated with this discrete-event simulation model?

(c) (5 pts.) List all the subroutines that you need in your simulation code.

(d) (5 pts.) Suppose that you simulated the system for τ minutes and recorded the process $\{L_i(t), 0 \le t \le \tau\}$, where $L_i(t)$ is the number of jobs at station $i \in \{1, 2\}$ at time t. Provide a point estimator for the average number of jobs in the system during the first τ minutes.

(e) (5 pts.) Use the recorded process $\{L_i(t), 0 \le t \le \tau\}$ from part **(d)** to provide an estimator for the average time both servers are busy during the first τ minutes.

Problem 2. (15 points) Suppose that using the simulation code discussed in Problem 1, you have performed 20 independent replications and used the resulting output to compute two t-based 90% confidence intervals (one for each station) on the average number of jobs at each station during the first τ minutes. Provide a bound on the probability that both confidence intervals include their true means. Explain your answer.

Problem 3. (25 points) Let X be a discrete random variable that is uniformly distributed over the set $\{a, a+1, \ldots, b\}$, where a and b are positive integers such that $a < b$.

(a) (9 pts.) Provide an algorithm that generates random variates for X.

(b) (9 pts.) Prove that the algorithm you provided in part **(a)** works.

(c) (7 pts.) Discuss the computational effort involved in the algorithm you provided in part **(a)**. In particular, what are the expected number of operations (addition/subtraction, division/multiplication, logarithms, comparisons, etc.) and random numbers needed to generate one random variate?

Problem 4. (35 points) Consider an M/G/1 queueing system that operates under the first-come-first-serve discipline. For this queueing system, let A_i denote the interarrival time between the ith and $(i-1)$th arriving customers for $i = 2, 3, \ldots$ and let A_1 denote the arrival time of the first customer. Let also S_i and W_i denote the service time and queue-waiting time for the ith customer, where $i = 1, 2, \ldots$. Assume that this queueing system is stable, i.e., its traffic intensity is less than one, and hence its long-run average waiting time in queue exists (denote it by ω). Assuming that the system is empty and idle at time zero, we have $W_1 = 0$ and

$$W_{i+1} = \max\{0, W_i + S_i - A_{i+1}\}, \text{ for } i = 1, 2, \ldots,$$

by Lindley's equation. Consider a single simulation run of the stochastic process $\{W_i, i = 1, 2, \ldots, n\}$, where n denotes the total number of customers that entered service during the simulation run.

(a) (10 pts.) Let $Q = \{i_1, i_2, \ldots, i_{20}\}$ be the set of customers that experienced no delay in queue during the given simulation run, i.e., $W_j = 0$ for all $j \in Q$. (Assume that the elements of Q are ordered such that $i_1 < i_2 < \cdots < i_{20}$ and $i_i = 1$.) Using this simulation run, provide a point estimator for the long-run average queue-waiting time ω by applying the regenerative method for simulation output analysis. In particular, use only the notation provided in this question to express your point estimator.

(b) (5 pts.) Discuss whether the point estimator you provided in part **(a)** is biased or not. Why or why not?

(c) (10 pts.) The busy period of a queueing system is defined as the period that begins with the time an incoming customer finds the system empty and ends the first time a departing customer leaves the system empty. Let X be a random variable that denotes the number of customers served during a busy period of the M/G/1 queue under consideration. Using the simulated sample path given in this question, provide a 95% confidence interval on $E[X]$.

(d) (5 pts.) Discuss whether the point estimator you provided in part **(c)**, i.e., the center of the confidence interval, is biased or not. Why or why not?

(e) (5 pts.) Discuss whether the confidence interval you provided in part **(c)** is approximate or not. Why or why not?